# 4. NONLINEAR STRUCTURAL ANALYSIS

**LINEAR FEM MODEL OF DEFORMABLE STRUCTURE BEHAVIOUR:**

$$[K]\{q\}=\{F\},$$

$$\{q\}=[K]^{-1}\{F\}.$$

Superposition method:

$$\{F_*\}=\alpha\{F_a\}+\beta\{F_b\}$$

$$\{q_*\}=[K]^{-1}\{F_*\},$$

$$\{q_*\}=[K]^{-1}(\alpha\{F_a\}+\beta\{F_b\})=\alpha[K]^{-1}\{F_a\}+\beta[K]^{-1}\{F_b\}$$

$$\{q_*\}=\alpha\{q_a\}+\beta\{q_b\}$$

Structural nonlinearities cause the response of a structure to vary disproportionately with the applied forces. Realistically, almost all structures are nonlinear in nature but not always to a degree that the nonlinearities have a significant effect on an analysis.

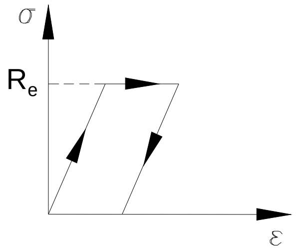
***NONLINEAR FEM MODEL***

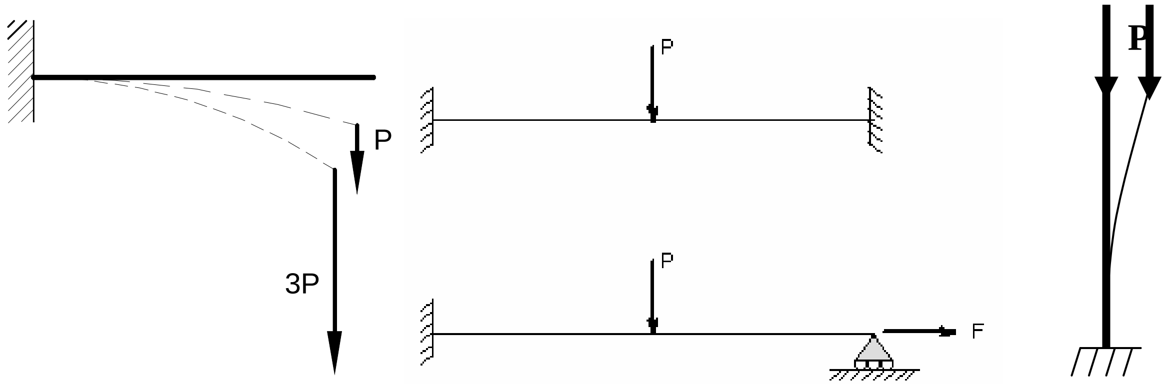
In a nonlinear analysis, the structure's stiffness matrix and load vector may depend on the solution and therefore are unknown. To solve the problem, the program uses an iterative procedure in which a series of linear approximations converges to the actual nonlinear solution.

$$\left[K\left(\{q\}\right)\right]\{q\}=\{F(q)\}.$$

- If the solution exists? How many solutions exist?
- Time consuming solution
- Iterative process of solution – problem of convergence
- Results of a load depend on loading history

# Causes of Nonlinear Behaviour




*simple elasto-plastic model of material behaviour*

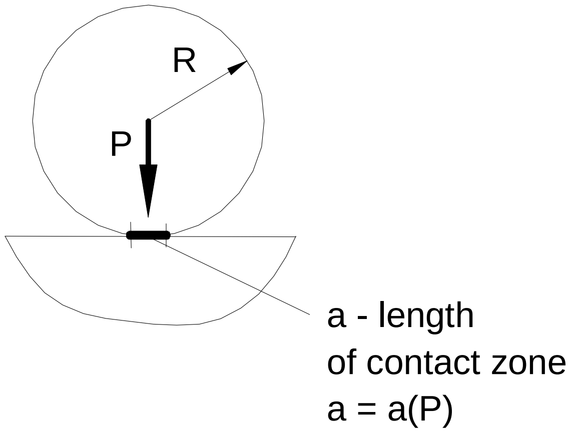
*large deformations of structures*

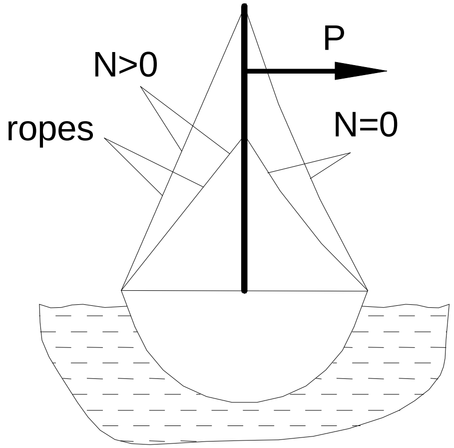
**Material Nonlinearities -** nonlinear stress – strain relationships. Many factors can influence a material's stress-strain properties, including load history (as in elasto-plastic response) environmental conditions (such as temperature), and the amount of the time that a load is applied (as in creep response)

**Geometric Nonlinearities.** If a structure experiences large deformations, its changing geometric configuration can cause the structure to respond nonlineary.

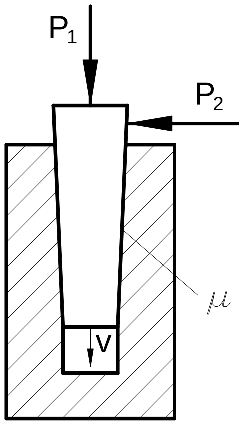
Under lateral loads, the beam is very flexible. As the force P increases, the rod deflects so much that the moment arm decreases appreciably, causing the increasing stiffness at higher loads. Axial forces can increase or decrease the stiffness of the beam depending oh the direction of the forces.

Friction, contact interaction, gaps, ropes

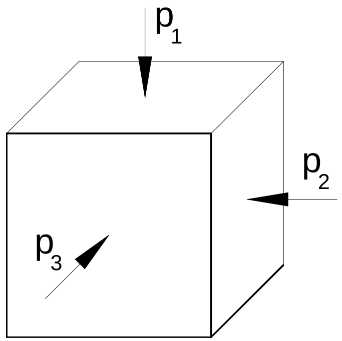



**Importance of the history of loading on the result of the final load**




friction




elasto-plastic material

## Iterative solution of a set of nonlinear simultaneous equations

The series of approximate solutions (iterations) $\{q\}_0$ , $\{q\}_1$, $\{q\}_2$ , ..., $\{q\}_n$

converging to the exact solution

.The vector $\{q\}_i$ is calculated on the base of the previous solution $\{q\}_{i-1}$ :

$$\left[K\left(\{q\}_{i-1}\right)\right]\{q\}_i = \{F\}.$$

$\{q\}_0$ – arbitrary initial solution (=0),

$$[K]_i = \left[K\left(\{q\}_i\right)\right].$$

**Convergence criteria**

DOF increment convergence $\{\Delta q\}_i = \{q\}_i - \{q\}_{i-1}$ ,

Out of balance convergence (residual vector)

$$\{R\}_{i+1} = \{F\} - [K]_i \{q\}_i ,$$

$$\left\|\{\Delta q\}_i\right\| \le \delta ,$$

$$\left\|\{R\}_i\right\| \le \varepsilon ,$$

$\delta$ i $\varepsilon$ - the reference values

Norms:

$$\left\|\{x\}\right\| = \left(\lfloor x \rfloor \{x\}\right)^{\frac{1}{2}} ,$$

$$\left\|\{x\}\right\| = \max x_i ,$$

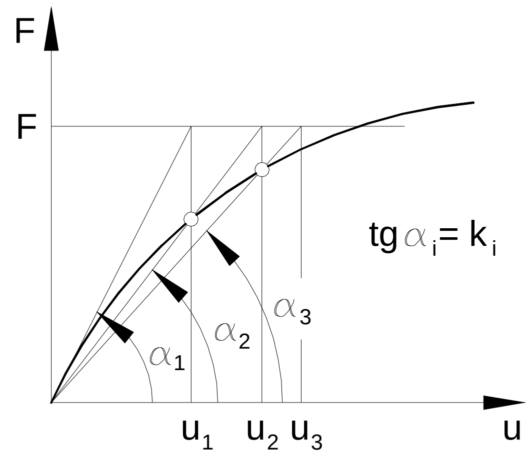
Criteria concerning the relative errors

$$\frac{\left\|\{\Delta q\}_i\right\|}{\left\|\{q\}_i\right\|} \le \varepsilon ,$$

$$\frac{\left\|\{R\}_i\right\|}{\left\|\{F\}\right\|} \le \delta .$$

**Direct approach**

$$\{q\}_i = [K]_{i-1}^{-1}\{F\}$$




**Incremental approach**

The calculations concern increments of the unknowns vector $\{q\}_i$

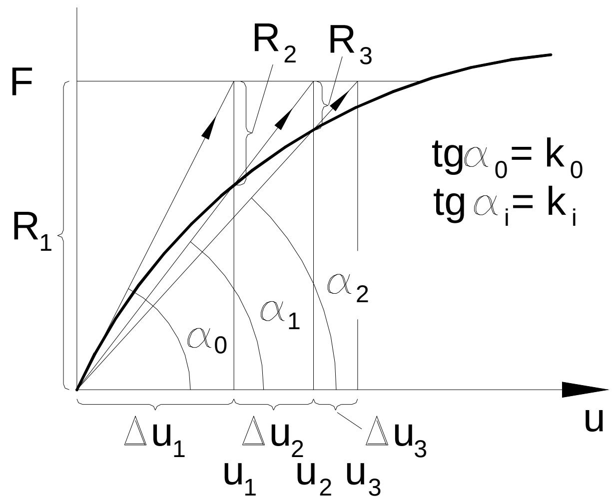
*Steps:*

$$\{R\}_i = \{F\} - [K]_{i-1}\{q\}_{i-1},$$

Increment

$$\{\Delta q\}_i = [K]_{i-1}^{-1}\{R\}_i .$$

Tne new approximate solution and the matrix



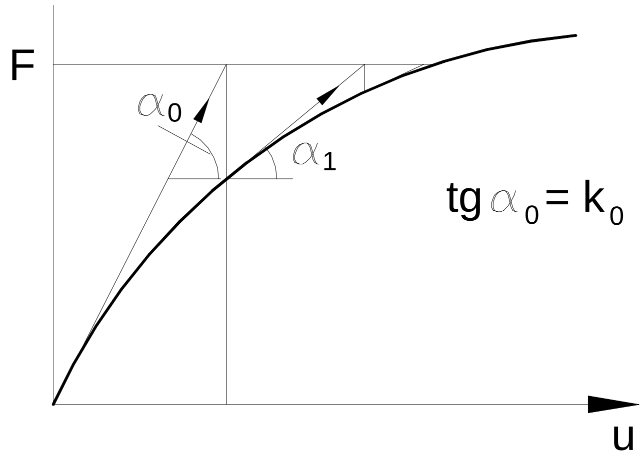
$$\{q\}_i = \{q\}_{i-1} + \{\Delta q\}_i \,,$$
$$[K]_i = \left[ K\left(\{q\}_i\right)\right],$$

**Newton-Raphson method**

In each itaration in the calculations of linear set of the equation uses the tangent matrix:

$$[K]_T = \frac{d\{F\}}{d\{q\}} = [K] + \frac{d[K]}{d\{q\}}\{q\}$$

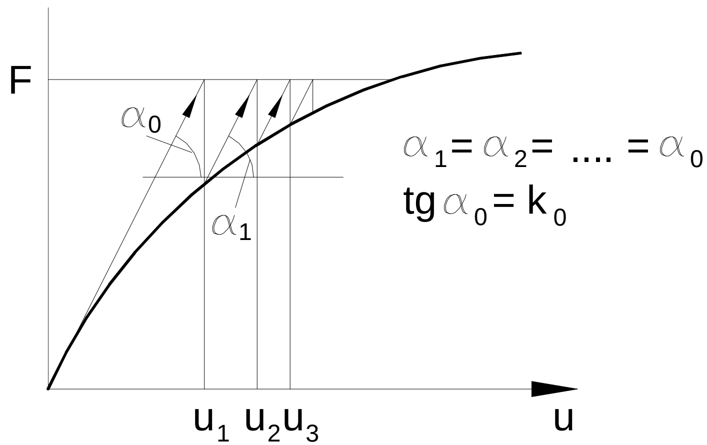
instead of coefficient matr



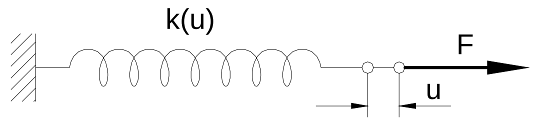

**Modified Newton–Raphson procedure**

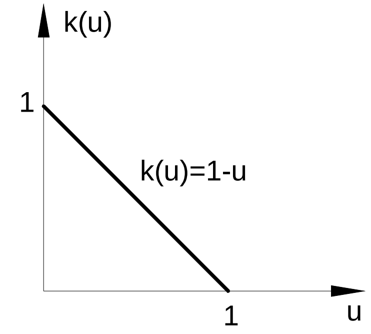
In each iteration the same set of the equations (the same initial matrix) are used

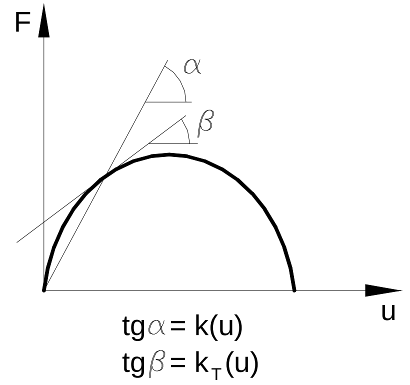
$$[K_0]^{-1} \text{ instead of } [K]_{i-1}^{-1}$$



**Example:**




Find the displacement u for the nonlinear spring:

$$k(u) = 1 - u$$

$$F = 0.2$$

**Analytical solution**

$$k(u)u = F\,, \quad u^2 - u + F = 0$$

$$u_1 = \frac{1-\sqrt{1-4F}}{2} = 0.2734,$$

$$u_2 = \frac{1+\sqrt{1-4F}}{2} = 0.7236.$$

.
tangent stiffness

$$k_T = \frac{dF}{du} = \frac{d}{du}\big(k(u)u\big) = \frac{dk}{du}u + k = 1 - 2u\,.$$

## Direct iteration procedure (the incremental approach

| $i$ | $u_{i-1}$ | $k_{i-1} = 1 - u_{i-1}$ | $R_i = F - k_{i-1}u_{i-1}$ | $\Delta u_i = R_i / k_{i-1}$ | $u_i = u_{i-1} + \Delta u_i$ | $\frac{\Delta u_i}{u_i}$ | $\frac{R_i}{F}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0.2 | 0.2 | 0.2 | 1 | 1 |
| 2 | 0.2 | 0.8 | 0.04 | 0.05 | 0.25 | 0.2 | 0.2 |
| 3 | 0.25 | 0.75 | 0.0125 | 0.0167 | 0.2667 | 0.063 | 0.063 |
| 4 | 0.2667 | 0.733 | 0.0044 | 0.006 | 0.2727 | 0.022 | 0.022 |
| 5 | 0.2727 | 0.7273 | 0.0017 | 0.0023 | 0.2750 | 0.008 | 0.0085 |

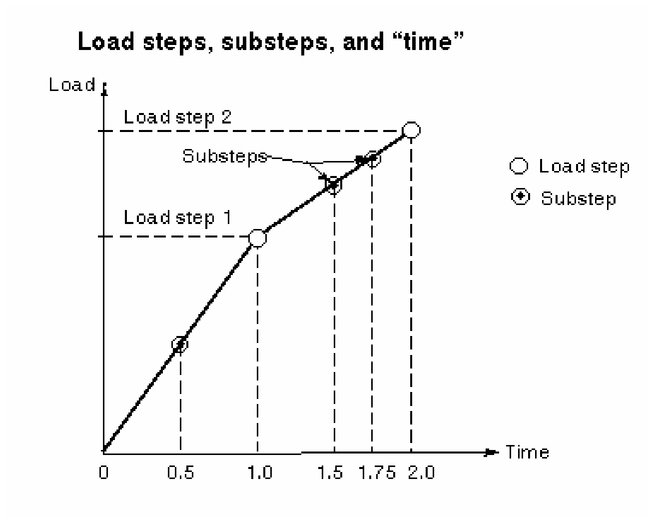
**Modified Newton-Raphson procedure**

| $i$ | $u_{i-1}$ | $k_{i-1} = 1 - u_{i-1}$ | $R_i = F - k_{i-1}u_{i-1}$ | $\Delta u_i = R_i / k_0$ | $u_i = u_{i-1} + \Delta u_i$ | $\frac{\Delta u_i}{u_i}$ | $\frac{R_i}{F}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0.2 | 0.2 | 0.2 | 1 | 1 |
| 2 | 0.2 | 0.8 | 0.04 | 0.04 | 0.24 | 0.167 | 0.2 |
| 3 | 0.24 | 0.76 | 0.0176 | 0.0176 | 0.2576 | 0.068 | 0.088 |
| 4 | 0.2576 | 0.7424 | 0.0087 | 0.00876 | 0.2664 | 0.033 | 0.044 |
| 5 | 0.2664 | 0.7336 | 0.0046 | 0.0046 | 0.2710 | 0.017 | 0.023 |
| 6 | 0.2710 | 0.729 | 0.0024 | 0.0024 | 0.2734 | 0.009 | 0.012 |

**Newton-Raphson procedure**

| $i$ | $u_{i-1}$ | $k_{i-1} = 1 - u_{i-1}$ | $R_i = F - k_{i-1}u_{i-1}$ | $k_{Ti} = 1 - 2u_{i-1}$ | $\Delta u_i = R_i / k_{Ti}$ | $u_i = u_{i-1} + \Delta u_i$ | $\frac{\Delta u_i}{u_i}$ | $\frac{R_i}{F}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0.2 | 1 | 0.2 | 0.2 | 1 | 1 |
| 2 | 0.2 | 0.8 | 0.04 | 0.6 | 0.0667 | 0.2667 | 0.250 | 0.2 |
| 3 | 0.2667 | 0.7333 | 0.0044 | 0.466 | 0.0095 | 0.2762 | 0.048 | 0.034 |
| 4 | 0.2762 | 0.7238 | 0.0001 | 0.448 | 0.0002 | 0.2764 | 0.001 | 0.0005 |

**Iterative nonlinear calculations in practice**




The user executes a nonlinear static analysis by subdividing the load into a series of incremental load steps and, at each step, performing a successive of linear approximations to obtain equilibrium.

Each linear approximation requires one pass through the equation solver (known as an equilibrium iteration).